Assignment 2

1. Let f and g be in $R_{2\pi}$. Their convolution (product) is defined to be

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y)dy.$$

Formally show the followings:

(a) $f * g$ belongs to $R_{2\pi}$.

(b) $g * f = f * g$.

(c) $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n), \quad \forall n \in \mathbb{Z}$.

It shows convolution is turned into pointwise product (of bisequences) under the Fourier transform.

2. Let $f \in R_{2\pi}$ and its primitive function be given by

$$F(x) = \int_0^x f(x)dx.$$

Show that F is 2π-periodic if and only if f has zero mean. In this case,

$$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.$$

3. Provide a proof of Theorem 1.6.

4. A function f defined on some $E \subset \mathbb{R}$ is called uniformly Hölder continuous with exponent $\alpha \in (0,1)$ if there exists some constant C such that $|f(x) - f(y)| \leq C|x-y|^\alpha$, $\forall x, y \in E$. It is called Lipschitz continuous when $\alpha = 1$. Show that for a uniformly Hölder or Lipschitiz continuous, 2π-periodic function, its Fourier coefficients satisfy

$$|a_n| \leq \frac{C\pi^\alpha}{n^\alpha}, \quad |b_n| \leq \frac{C\pi^\alpha}{n^\alpha}.$$

5. Provide a proof of Theorem 1.5 when the Lipschitz condition is replaced by a Hólder condition.

6. Propose a definition for $\sqrt{d/dx}$. This operator should be a linear map which maps smooth functions to smooth functions and satisfy

$$\sqrt{\frac{d}{dx}}\sqrt{\frac{d}{dx}}f = f,$$

for all smooth, 2π-periodic f.

7. Establish the following two formulas:

(a)

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

(b)

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Hint: Examine the Fourier series of the functions $f_j, j = 1, ..4$, in Section 1.

8. Show that

$$\frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad \forall x \in [0, 2\pi]$$

and deduce

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

9. (a) Show that the Fourier series of the function $\cos tx$, $x \in [-\pi, \pi]$ where t is not an integer is given by

$$\frac{\pi \cos tx}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2} (-1)^n \cos nx, \quad x \in [-\pi.\pi].$$

(b) Deduce that for $t \in (0, 1)$,

$$\log \sin t\pi = \log t\pi + \sum_{n=1}^{\infty} \log \left(1 - \frac{t^2}{n^2}\right).$$

(c) Conclude that

$$\frac{\sin t\pi}{\pi} = t \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{n^2}\right), \quad t \in (0, 1).$$

10. Here is an interesting application of Property III.

(a) Show that $\dfrac{1}{\sin x/2} - \dfrac{2}{x}$ is bounded on $(0, \pi)$.

(b) Show that

$$\lim_{n\to\infty} \int_0^{\pi} \left(\frac{1}{\sin x/2} - \frac{2}{x}\right) \sin(n + \frac{1}{2})x dx = 0.$$

(c) Show that

$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin(n + 1/2)x}{x} dx = \frac{\pi}{2}.$$

(d) Finally, deduce that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

The following problems are optional.

11. A sequence $\{x_n\}$ is convergent to x in arithmetic mean if $y_n = (x_0 + \cdots + x_{n-1})/n$ converges to x as $n \to \infty$. Show that $\{x_n\}$ converges to x in arithmetic mean when $\{x_n\}$ converges to x. However, give an example to show that the converse may be not true.

12. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

and

$$\sigma_n(x) = \frac{D_0(x) + \cdots + D_{n-1}(x)}{n}.$$

Establish the formula

$$\sigma_n(x) = \frac{1}{\pi n} \int_{-\pi}^{\pi} \frac{\sin^2(nz/2)}{2\sin^2 z/2} f(x + z) dz, \quad n \geq 1.$$

13. Let s_n be the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$. The series is called convergent in arithmetic mean to s if $\{s_n\}$ converges to s in arithmetic mean. Show that for every 2π-periodic function integrable on $[-\pi, \pi]$, its Fourier series converges in arithmetic mean to $f(x)$ where x is a point of continuity and to $\big(f(x^+) + f(x^-)\big)/2$ where x is a jump discontinuity.